42nd International Mathematical Olympiad

IMO 2001 Problems and Solutions

Washinton, DC, USA
July 1-14, 2001

Problem 1

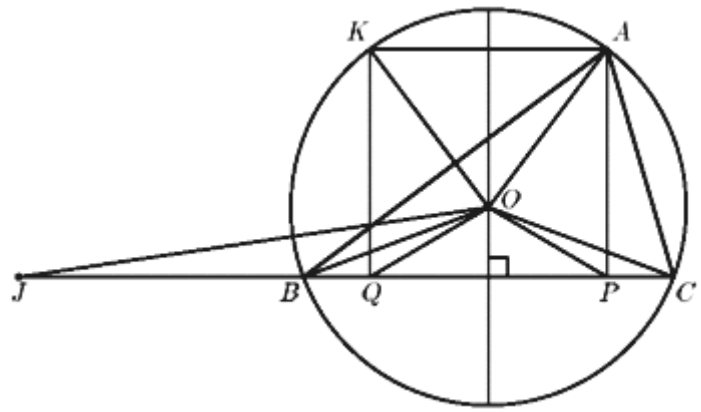
Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA >= \angle ABC + 30°$.

Prove that $\angle CAB + \angle COP < 90°$.

Solution 1

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and $\delta = \angle COP$. Let K and Q be the reflections of A and P, respectively, across the perpendicular bisector of BC. Let R denote the circumradius of $\triangle ABC$. Then $OA = OB = OC = OK = R$. Furthermore, we have $QP = KA$ because $KQPA$ is a rectangle. Now note that $\angle AOK = \angle AOB - \angle KOB = \angle AOB - \angle AOC = 2\gamma - 2\beta >= 60°$.



It follows from this and from $OA = OK = R$ that $KA >= R$ and $QP >= R$. Therefore, using the Triangle Inequality, we have $OP + R = OQ + OC > QC = QP + PC >= R + PC$. It follows that $OP > PC$, and hence in ΔCOP, $\angle PCO > \delta$. Now since $\alpha = 1\backslash 2\ \angle BOC = 1\backslash 2\ (180° - 2\angle PCO) = 90° - \angle PCO$, it indeed follows that $\alpha + \delta < 90°$.

Solution 2

As in the previous solution, it is enough to show that $OP > PC$. To this end, recall that by the (Extended) Law of Sines, $AB = 2R\sin\gamma$ and $AC = 2R\sin\beta$. Therefore, we have $BP - PC = AB\cos\beta - AC\cos\gamma = 2R(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 2R\sin(\gamma - \beta)$. It follows from this and from

$30° <= \gamma - \beta < \gamma < 90°$

that $BP - PC >= R$. Therefore, we obtain that $R + OP = BO + OP > BP >= R + PC$, from which $OP > OC$, as desired.

Solution 3

We first show that $R^2 > CP \cdot CB$. To this end, since $CB = 2R\sin\alpha$ and $CP = AC\cos\gamma = 2R\sin\beta\cos\gamma$, it suffices to show that $1\backslash 4 > \sin\alpha\sin\beta\cos\gamma$. We note that $1 > \sin\alpha = \sin(\gamma + \beta) = \sin\gamma\cos\beta + \sin\beta\cos\gamma$ and $1\backslash 2 <= \sin(\gamma - \beta) = \sin\gamma\cos\beta - \sin\beta\cos\gamma$ since $30° <= \gamma - \beta < 90°$. It follows that $1\backslash 4 > \sin\beta\cos\gamma$ and that $1\backslash 4 > \sin\alpha\sin\beta\cos\gamma$.

Now we choose a point J on BC so that $CJ \cdot CP = R^2$. It follows from this and from $R^2 > CP \cdot CB$ that $CJ > CB$, so that $\angle OBC > \angle OJC$. Since $OC/CJ = PC/CO$ and $\angle JCO = \angle OCP$, we have $\triangle JCO \cong \triangle OCP$ and $\angle OJC = \angle POC = \delta$. It follows that $\delta < \angle OBC = 90° - \alpha$ or $\alpha + \delta < 90°$.

Solution 4

On the one hand, as in the third solution, we have $R^2 > CP \cdot CB$. On the other hand, the power of P with respect to the circumcircle of $\triangle ABC$ is $BP \cdot PC = R^2 - OP^2$. From these two equations we find that

$$OP^2 = R^2 - BP \cdot PC > PC \cdot CB - BP \cdot PC = PC^2,$$

from which $OP > PC$. Therefore, as in the first solution, we conclude that $\alpha + \delta < 90°$.

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Problem 2

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1$$

for all positive real numbers a, b and c.

Solution

First we shall prove that

$$\frac{a}{\sqrt{a^2+8bc}}\geq\frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$

or equivalently, that

$$\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2\geq a^{\frac{2}{3}}(a^2+8bc).$$

The AM-GM inequality yields

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2-\left(a^{\frac{4}{3}}\right)^2 &= \left(b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\\ &\geq 2b^{\frac{2}{3}}c^{\frac{2}{3}}\cdot 4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}\\ &= 8a^{\frac{2}{3}}bc.\end{aligned}$$

Thus

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 &\geq \left(a^{\frac{4}{3}}\right)^2+8a^{\frac{2}{3}}bc\\ &= a^{\frac{2}{3}}(a^2+8bc),\end{aligned}$$

so

$$\frac{a}{\sqrt{a^2+8bc}}\geq\frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Similarly, we have

$$\frac{b}{\sqrt{b^2+8ca}}\geq\frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}\quad\text{and}$$

$$\frac{c}{\sqrt{c^2+8ab}}\geq\frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Adding these three inequalities yields

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Comment. The proposer conjectures that for any a, b, $c > 0$ and $\lambda >= 8$, the following inequality holds:

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ca}}+\frac{c}{\sqrt{c^2+\lambda ab}}\geq\frac{3}{\sqrt{1+\lambda}}.$$

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Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Solution 1

We introduce the following symbols: G is the set of girls at the competition and B is the set of boys, P is the set of problems, $P(g)$ is the set of problems solved by $g \in G$, and $P(b)$ is the set of problems solved by $b \in B$. Finally, $G(p)$ is the set of girls that solve $p \in P$ and $B(p)$ is the set of boys that solve p. In terms of this notation, we have that for all $g \in G$ and $b \in B$,

$$(a)\ \ |P(g)| \le 6, \quad |P(b)| \le 6, \qquad (b)\ P(g) \cap P(b) \ne \emptyset.$$

We wish to prove that some $p \in P$ satisfies $|G(p)| >= 3$ and $|B(p)| >= 3$. To do this, we shall assume the contrary and reach a contradiction by counting (two ways) all ordered triples (p, q, r) such that $p \in P(g) \cap P(b)$. With $T = \{(p, g, b) : p \in P(g) \cap P(b)\}$, condition (b) yields

$$|T| = \sum_{g \in G} \sum_{b \in B} |P(g) \cap P(b)| \ge |G| \cdot |B| = 21^2. \tag{1}$$

Assume that no $p \in P$ satisfies $|G(p)| >= 3$ and $|B(p)| >= 3$. We begin by noting that

$$\sum_{p \in P} |G(p)| = \sum_{g \in G} |P(g)| \le 6|G| \quad \text{and} \quad \sum_{p \in P} |B(p)| \le 6|B|. \tag{2}$$

(*Note*. The equality in (2) is obtained by a standard double-counting technique: Let $\chi(g, p) = 1$ if g solves p and $\chi(g, p) = 0$ otherwise, and interchange the orders of summation in $\Sigma_{p \in P} \Sigma_{g \in G} \chi(g, p)$.) Let

$$P_+ = \{p \in P : |G(p)| \ge 3\},$$
$$P_- = \{p \in P : |G(p)| \le 2\}.$$

Claim. $\Sigma_{p \in P-} |G(p)| >= |G|$ thus $\Sigma_{p \in P+} |G(p)| <= 5\,|G|$. Also $\Sigma_{p \in P+} |B(p)| >= |B|$ thus $\Sigma_{p \in P-} |B(b)| <= 5\,|B|$.

Proof. Let $g \in G$ be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that g solves some problem p that is solved by at least $\lceil 21/6 \rceil = 4$ boys. By assumption, $|B(p)| >= 4$ implies that $p \in P_-$, so every girl solves at least one problem in P_-. Thus

$$\sum_{p \in P_-} |G(p)| \ge |G|. \tag{3}$$

In view of (2) and (3) we have

$$\sum_{p \in P_+} |G(p)| = \sum_{p \in P} |G(p)| - \sum_{p \in P_-} |G(p)| \le 5|G|.$$

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a problem $p \in P_+$. Thus $\Sigma_{p \in P+} |B(p)| >= |B|$, and the calculation proceeds as before using (2). □

Using the claim just established, we find

$$
\begin{aligned}
|T| &= \sum_{p\in P} |G(p)|\cdot|B(p)| \\
&= \sum_{p\in P_+} |G(p)|\cdot|B(p)| + \sum_{p\in P_-} |G(p)|\cdot|B(p)| \\
&\le 2\sum_{p\in P_+} |G(p)| + 2\sum_{p\in P_-} |B(p)| \\
&\le 10|G| + 10|B| = 20\cdot 21.
\end{aligned}
$$

This contradicts (1), so the proof is complete.

Solution 2

Let us use some of the notation given in the first solution. Suppose that for every $p \in P$ either $|G(p)| <= 2$ or $|B(p)| <= 2$. For each $p \in P$, color p red if $|G(p)| <= 2$ and otherwise color it black. In this way, if p is red then $|G(p)| <= 2$ and if p is black then $|B(p)| <= 2$. Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, color the square corresponding to (g, b) as follows: pick $p \in P(g) \cap P(b)$ and assign p's color to that square. (By condition (b), there is always an available choice.) By the Pigeonhole Principle, one of the two colors is assigned to at least $\lceil 441/2 \rceil = 221$ squares, and thus some row has at least $\lceil 221/21 \rceil = 11$ black squares or some column has at least 11 red squares.

Suppose the row corresponding to $g \in G$ has at least 11 black squares. Then for each of 11 squares, the black problem that was chosen in assigning the color was solved by at most 2 boys. Thus we account for at least $\lceil 11/2 \rceil = 6$ distinct problems solved by g. In view of condition (a), g solves only these problems. But then at most 12 boys solve a problem also solved by g, in violation of condition (b).

In exactly the same way, a contradiction is reached if we suppose that some column has at least 11 red squares. Hence some $p \in P$ satisfies $|G(p)| >= 3$ and $|B(p)| >= 3$.

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Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i .$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Solution

Let $\Sigma S(a)$ be the sum of $S(a)$ over all $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$. We compute $\Sigma S(a)$ mod $n!$ two ways, one of which depends on the desired conclusion being false, and reach a contradiction when n is odd.

First way. In $\Sigma S(a)$, k_1 is multiplied by each $i \in \{1, \ldots, n\}$ a total of $(n - 1)!$ times, once

for each permutation of {1, ..., n} in which $a_1 = i$. Thus the coefficient of k_1 in $\Sigma S(a)$ is

$$(n-1)!\,(1+2+\cdots+n) = (n+1)!/2.$$

The same is true for all k_i, so

$$\sum S(a) = \frac{(n+1)!}{2}\sum_{i=1}^{n} k_i. \tag{1}$$

Second way. If $n!$ is not a divisor of $S(a) - S(b)$ for any $a \neq b$, then each S(a) must have a different remainder mod $n!$. Since there are $n!$ permutations, these remainders must be precisely the numbers 0, 1, 2, ..., $n! - 1$. Thus

$$\sum S(a) \equiv \frac{(n!-1)\,n!}{2} \mod n!\,. \tag{2}$$

Combining (1) and (2), we get

$$\frac{(n+1)!}{2}\sum_{i=1}^{n} k_i \equiv \frac{(n!-1)\,n!}{2} \mod n!\,. \tag{3}$$

Now, for n odd, the left side of (3) is congruent to 0 modulo $n!$, while for $n > 1$ the right side is not congruent to 0 ($n! - 1$ is odd). For $n > 1$ and odd, we have a contradiction.

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Problem 5

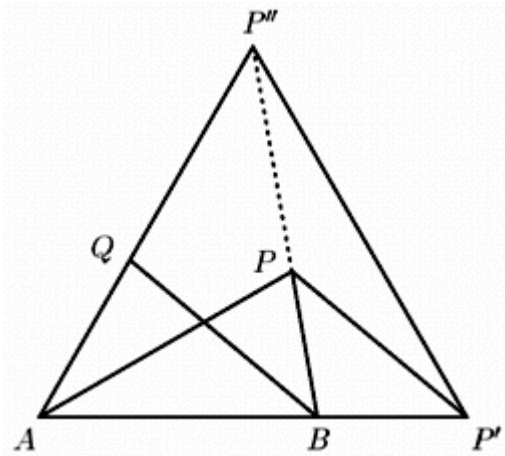
In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60°$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

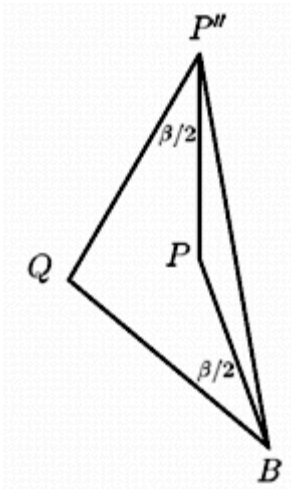
Solution

Denote the angles of ABC by $\alpha = 60°$, β, and γ. Extend AB to P' so that $BP' = BP$, and construct P'' on AQ so that $AP'' = AP'$. Then $BP'P$ is an isosceles triangle with base angle $\beta/2$. Since $AQ + QP'' = AB + BP' = AB + BP = AQ + QB$, it follows that $QP'' = QB$. Since $AP'P''$ is equilateral and AP bisects the angle at A, we have $PP' = PP''$.



Claim. Points *B*, *P*, *P''* are collinear, so *P''* coincides with *C*.

Proof. Suppose to the contrary that *BPP''* is a nondegenerate triangle. We have that $\angle PBQ = \angle PP'B = \angle P\,P''Q = \beta/2$. Thus the diagram appears as below, or else with *P* is on the other side of *BP''*. In either case, the assumption that *BPP''* is nondegenerate leads to *BP* = *PP''* = *PP'*, thus to the conclusion that *BPP'* is equilateral, and finally to the absurdity $\beta/2 = 60°$ so $\alpha + \beta = 60° + 120° = 180°$.



Thus points *B*, *P*, *P''* are collinear, and *P''* = *C* as claimed. □

Since triangle BCQ is isosceles, we have $120° - \beta = \gamma = \beta/2$, so $\beta = 80$ and $\gamma = 40°$. Thus *ABC* is a 60-80-40 degree triangle.

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Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$ac + bd = (b + d + a - c)(b + d - a + c)$.

Prove that $ab + cd$ is not prime.

Solution 1

Suppose to the contrary that $ab + cd$ is prime. Note that

$$ab + cd = (a + d)c + (b - c)a = m \cdot \gcd(a + d, b - c)$$

for some positive integer m. By assumption, either $m = 1$ or $\gcd(a + d, b - c) = 1$. We consider these alternatives in turn.

Case (i): $m = 1$. Then

$$\begin{aligned}\gcd(a+d,\, b-c) &= ab+cd > ab+cd-(a-b+c+d)\\ &= (a+d)(c-1)+(b-c)(a+1)\\ &\geq \gcd(a+d,\, b-c),\end{aligned}$$

which is false.

Case (ii): gcd(a + d, b - c) = 1. Substituting $ac + bd = (a + d)b - (b - c)a$ for the left-hand side of $ac + bd = (b + d + a - c)(b + d - a + c)$, we obtain

$$(a+d)(a-c-d) = (b-c)(b+c+d).$$

In view of this, there exists a positive integer k such that

$$\begin{aligned}a-c-d &= k(b-c),\\ b+c+d &= k(a+d).\end{aligned}$$

Adding these equations, we obtain $a + b = k(a + b - c + d)$ and thus $k(c - d) = (k - 1)(a + b)$. Recall that $a > b > c > d$. If $k = 1$ then $c = d$, a contradiction. If k >= 2 then

$$2 \geq \frac{k}{k-1} = \frac{a+b}{c-d} > 2,$$

a contradiction.

Since a contradiction is reached in both (i) and (ii), $ab + cd$ is not prime.

Solution 2

The equality $ac + bd = (b + d + a - c)(b + d - a + c)$ is equivalent to

$$a^2 - ac + c^2 = b^2 + bd + d^2. \tag{1}$$

Let $ABCD$ be the quadrilateral with $AB = a$, $BC = d$, $CD = b$, $AD = c$, $\angle BAD = 60°$, and $\angle BCD = 120°$. Such a quadrilateral exists in view of (1) and the Law of Cosines; the common value in (1) is BD^2. Let $\angle ABC = \alpha$, so that $\angle CDA = 180° - \alpha$. Applying the Law of Cosines to triangles ABC and ACD gives

$$a^2 + d^2 - 2ad\cos\alpha = AC^2 = b^2 + c^2 + 2bc\cos\alpha.$$

Hence $2\cos\alpha = (a^2 + d^2 - b^2 - c^2)/(ad + bc)$, and

$$AC^2 = a^2 + d^2 - ad\,\frac{a^2 + d^2 - b^2 - c^2}{ad + bc} = \frac{(ab + cd)(ac + bd)}{ad + bc.}$$

Because *ABCD* is cyclic, Ptolemy's Theorem gives

$$(AC \cdot BD)^2 = (ab + cd)^2$$

It follows that

$$(ac + bd)(a^2 - ac + c^2) = (ab + cd)(ad + bc). \tag{2}$$

(*Note*. Also straightforward algebra can be used obtain (2) from (1).) Next observe that

$$ab + cd > ac + bd > ad + bc. \tag{3}$$

The first inequality follows from $(a - d)(b - c) > 0$, and the second from $(a - b)(c - d) > 0$.

Now assume that $ab + cd$ is prime. It then follows from (3) that $ab + cd$ and $ac + bd$ are relatively prime. Hence, from (2), it must be true that $ac + bd$ divides $ad + bc$. However, this is impossible by (3). Thus $ab + cd$ must not be prime.

Note. Examples of 4-tuples (a, b, c, d) that satisfy the given conditions are (21, 18, 14, 1) and (65, 50, 34, 11).