2008 IMO Problems/Problem 2

From AoPSWiki

Problem 2

(i) If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$. (With the $\sum$ sign for cyclic summation, this inequality could be rewritten as $\sum \frac{x^2}{(x-1)^2} \geq 1$.)

(ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

Solution

Consider the transormation $f : \mathbb{R}/\{1\} \to \mathbb{R}/\{-1\}$ defined by $f(u) = \frac{u}{1-u}$ and put $\alpha = f(x), \beta = f(y), \gamma = f(z)$. Since f is also one-to one from $\mathbb{Q}/\{1\}$ to $\mathbb{Q}/\{-1\}$, the problem is equivalent to showing that $\alpha^2 + \beta^2 + \gamma^2 \geq 1 \quad (1)$ subject to $\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{\beta}{\beta+1}\right)\left(\frac{\gamma}{\gamma+1}\right) = 1 \quad (2)$ and that equallity holds for infinitely many triplets of *rational* α, β, γ.

Now, rewrite (2) as $\alpha\beta\gamma = (1+\alpha)(1+\beta)(1+\gamma)$ and express it as $0 = 1 + p + q$ where $p = \alpha + \beta + \gamma$ and $q = \alpha\beta + \beta\gamma + \gamma\alpha$. Notice that (1) can be written as $p^2 - 2q \geq 1$. But from $p = -1 - q$, we get $p^2 - 2q = (1+q)^2 - 2q = 1 + q^2 \geq 1$, with equality holding iff $q = 0$. That proves part **(i)** and points us in the direction of looking for rational α, β, γ for which $q = 0$ and (hence) $p = -1$, that is:

$\alpha + \beta + \gamma = -1$

$\alpha\beta + \beta\gamma + \gamma\alpha = 0$ Expressing α from the first equation and substituting into the second, we get $\beta\gamma + (\beta+\gamma)(-1-\beta-\gamma) = 0$ as the sole condition we need to satisfy in rational numbers.

If $\beta = \frac{b}{m}$ and $\gamma = \frac{c}{m}$ for some integers b, c, and m, they would need to satisfy $bc = m(b+c) + (b+c)^2 \Leftrightarrow m = \frac{bc}{b+c} - (b+c)$. For m to be integer, we would like $b+c$ to divide bc. Consider the example $b = t, c = t^2 - t, m = t - 1 - t^2$, where

$b+c=t^2$divides $bc=t(t^2-t)$for any integer $t\neq 0$. Substituting back, that gives us $\beta=\dfrac{t}{t-1-t^2},\quad \gamma=\dfrac{t^2-t}{t-1-t^2},\quad \alpha=\dfrac{1-t}{t-1-t^2}$. A simple check shows that $\alpha,\beta,\gamma$are rational and well defined and that $p=-1$and $q=0$for *any* integer t(even for $t=0$).

Moreover, from $\lim_{t\to+\infty}\beta=0$and $\beta<0$for large t, we see that infinitely many t generate infinitely many *different* triplets of α, β, and γ. That completes the proof of part **(ii)**. --Vbarzov 03:03, 5 September 2008 (UTC)

2008 IMO Problems/Problem 3

From AoPSWiki

Problem

Prove that there are infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Solution

The main idea is to take a gaussian prime $a+bi$ and multiply it by a "twice as small" $c+di$ to get $n+i$. The rest is just making up the little details.

For each *sufficiently large* prime p of the form $4k+1$, we shall find a corresponding n such that p divides n^2+1 and $p>2n+\sqrt{2n}$. Since there exist infinitely many such primes and, for each of them, $n\geq\sqrt{p-1}$, we will have found infinitely many *distinct* n satisfying the hypothesis.

Take a prime p of the form $4k+1$ and consider its "sum-of-two squares" representation $p=a^2+b^2$, which we know to exist for all such primes. As $a\neq b$, assume without loss of generality that $b>a$. If $a=1$, then $n=b$ is what we are looking for, and $p=n^2+1>2n+\sqrt{2n}$ as long as p (and hence n) is large enough. Assume from now on that $b>a>1$.

Since a and b are (obviously) co-prime, there must exist integers c and d such that $ad+bc=1.\quad(1)$ In fact, if c and d are such numbers, then $c\pm ma$ and $d\mp mb$ work as well for any integer m, so we can assume that $c\in\left[-\frac{a}{2},\frac{a}{2}\right]$.

Define $n=|ac-bd|$ and let's see why this is a good choice. For starters, notice that $(a^2+b^2)(c^2+d^2)=n^2+1$.

If $c=\pm\frac{a}{2}$, from (1) we see that a must divide 2 and hence $a=2$. This implies, $d=-\frac{b-1}{2}$ and $n=\frac{b(b-1)}{2}-2$. Therefore, $\left(b-\frac{1}{2}\right)^2=1/4+2(n+2)>2n$, so $b>\sqrt{2n}+\frac{1}{2}$. Finally, $p=b^2+2^2>2n+\sqrt{2n}$ and the case $c=\pm\frac{a}{2}$ is cleared.

We can safely assume now that $|c| \leq \frac{a-1}{2}$. As $b > a > 1$ implies $b > 2$, we have $|d| = \left|\frac{1-bc}{a}\right| \leq \frac{b(a-1)+2}{2a} < \frac{ba}{2a} = \frac{b}{2}$, so $|d| \leq \frac{b-1}{2}$.

Therefore,
$$n^2 + 1 = (a^2 + b^2)(c^2 + d^2) \leq p\left(\frac{(a-1)^2}{4} + \frac{(b-1)^2}{4}\right). \quad (2)$$

Before we proceed, we would like to show first that $a + b - 1 > \sqrt{p}$. Observe that the function $x + \sqrt{p - x^2}$ over $x \in (2, \sqrt{p-4})$ reaches its minima on the ends, so $a + b$ given $a^2 + b^2 = p$ is minimized for $a = 2$, where it equals $2 + \sqrt{p - 2^2}$. So we want to show that $2 + \sqrt{p-4} > \sqrt{p} + 1$, which is not hard to show for *large enough* p.

Now armed with $a + b - 1 > \sqrt{p}$ and (2), we get
$4(n^2 + 1) \leq p(a^2 + b^2 - 2(a + b - 1))$
$< p(p - 2\sqrt{p})$
$< u^2(u-1)^2,$ where $u = \sqrt{p}$.

$$u^2(u-1)^2 > 4n^2 + 4 > 4n^2 \Rightarrow$$
$$u(u-1) > 2n \Rightarrow$$
$$(u - \frac{1}{2})^2 > 2n + \frac{1}{4} > 2n \Rightarrow$$
$$u > \sqrt{2n} + \frac{1}{2} \Rightarrow$$

Finally, $p = u^2 > 2n + \sqrt{2n}$. The proof is complete.--Vbarzov 03:02, 5 September 2008 (UTC)

2008 IMO Problems/Problem 4

From AoPSWiki

Problem

Find all functions $f:(0,\infty)\mapsto(0,\infty)$(so $f$is a function from the positive real numbers) such that

$$\frac{(f(w))^2+(f(x))^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers w,x,y,z,satisfying $wx=yz.$

Solution

Considering $w=1$and $z=y=\sqrt{x}$which satisfy the constraint $wx=yz$we get the following equation:

$$\frac{(f(1))^2+(f(x))^2}{f(x)+f(x)}=\frac{1+x^2}{x+x}\Leftrightarrow x((f(1))^2+(f(x))^2)=(1+x^2)f(x)$$

At once considering $x=1$we get $(f(1))^2=f(1)$and knowing that $f:\mathbb{R}^+\to\mathbb{R}^+$the only possible solution is $f(1)=1$since $f(1)=0$is impossible.

So we get the quadratic equation:

$$x(f(x))^2-(1+x^2)f(x)+x=0$$

Solving for $f(x)$as a function of $x$we get:

$$f(x) = \frac{1 + x^2 \pm \sqrt{(1+x^2)^2 - 4x^2}}{2x} = \frac{1 + x^2 \pm \sqrt{(1-x^2)^2}}{2x}$$

At once we see that for one value of x, $f(x)$can only take one of 2 possible values:

$$f(x) = x \vee f(x) = \frac{1}{x}.$$

Take into consideration that $f(2) = 2$but $f(3) = \frac{1}{3}$verifies the quadratic equation and thus so far we can't say that $f(x) = x \, \forall_{x \in \mathbb{R}^+}$or alternatively $f(x) = \frac{1}{x} \, \forall_{x \in \mathbb{R}^+}$. This is indeed the case but we haven't proved it yet.

To prove the previous assertion consider 2 values $a, b \in \mathbb{R}^+$such that $a \neq 1 \wedge b \neq 1 \wedge a \neq b$while having $f(a) = a \wedge f(b) = \frac{1}{b}$

Consider now the original functional equation with $w = a, \; x = b, \; y = z = \sqrt{ab}$which verifies the constraint. Substituting we have:

$$\frac{(f(a))^2 + (f(b))^2}{f(ab) + f(ab)} = \frac{a^2 + b^2}{ab + ab} \Leftrightarrow f(ab) = ab\frac{a^2 + \frac{1}{b^2}}{a^2 + b^2}$$

Now either $f(ab) = ab$or $f(ab) = \frac{1}{ab}$. (notice that $ab \neq b \wedge ab \neq b$by hypothesis)

If $f(ab) = ab$ then we have $ab = ab\frac{a^2 + \frac{1}{b^2}}{a^2 + b^2} \Leftrightarrow b^4 = 1$ and since $b > 0$ the only solution is $b = 1$.

If $f(ab) = \frac{1}{ab}$ then we have $\frac{1}{ab} = ab\frac{a^2 + \frac{1}{b^2}}{a^2 + b^2} \Leftrightarrow a^2 + b^2 = a^4b^2 + a^2 \Leftrightarrow a^4 = 1$ and since $a > 0$ the only solution is $a = 1$.

So the only solutions are $a = 1$ or $b = 1$ in which case both alternatives imply $f(1) = 1$.
Thus we conclude that solutions to the functional equation are a subset of
$\left\{ f(x) = x \ \forall_{x \in \mathbb{R}^+}, \ f(x) = \frac{1}{x} \ \forall_{x \in \mathbb{R}^+} \right\}$.

Finally plug each of these 2 functions into the functional equation and verify that they indeed are solutions.

This is trivial since $f(x) = x$ is an obvious solution and for $f(x) = \frac{1}{x}$ we have:

$\frac{\frac{1}{w^2} + \frac{1}{x^2}}{\frac{1}{y^2} + \frac{1}{z^2}} = \frac{\frac{w^2+x^2}{(wx)^2}}{\frac{y^2+z^2}{(yz)^2}} = \frac{w^2 + x^2}{y^2 + z^2}$ provided that $(wx)^2 = (yz)^2$ which verifies the original constraint.

So the functional equation has 2 solutions:

$$f(x) = x \ \forall_{x \in \mathbb{R}^+} \ \vee \ f(x) = \frac{1}{x} \ \forall_{x \in \mathbb{R}^+}$$

2008 IMO Problems/Problem 5

From AoPSWiki

Problem 5

Let $n$and $k$be positive integers with $k \geq n$and $k - n$an even number. Let $2n$lamps labelled 1, 2, ..., $2n$be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let Nbe the number of such sequences consisting of $k$steps and resulting in the state where lamps 1through $n$are all on, and lamps $n+1$through $2n$are all off.

Let Mbe number of such sequences consisting of $k$steps, resulting in the state where lamps 1through $n$are all on, and lamps $n+1$through $2n$are all off, but where none of the lamps $n+1$through $2n$is ever switched on.

Determine $\frac{N}{M}$.

Solution

For convenience, let Adenote the set $(1, 2, \ldots n)$and $B$the set $(n+1, n+2, \ldots, 2n)$.

We can describe each sequences of switching the lamps as a k-dimensional vector $(a_1, a_2, \ldots, a_k)$, where $a_i \in A \cup B$signifies which lamp was switched on the i-th move for $i = 1, 2, \ldots k$.

Let $\mathcal{N}$consist of those sequences that contain each of the numbers in Aan *odd* number of times and each of the numbers in Ban *even* number of times. Similarly, let $\mathcal{M}$denote the set of those sequences that contain no numbers from $B$and each of the numbers in Aan odd number of times. By definition, $M = |\mathcal{M}|$and $N = |\mathcal{N}|$.

Define the mapping $f : \mathcal{N} \to \mathcal{M}$as

$$f(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots b_k) : b_i = \begin{cases} a_i, & \text{if } a_i \in A \\ a_i - n, & \text{if } a_i \in B \end{cases}$$

What we want to show now is that each element of $\mathcal{M}$is an image of exactly 2^{k-n} elements from $\mathcal{N}$, which would imply $N = 2^{k-n}M$and solve the problem.

Consider an arbitrary element $y$of $\mathcal{M}$and let $l_i$be the number of appearances of the number $i$in $y$for $i = 1, 2, \ldots n$. Now consider the set of pre-images of y, that is $X_y = \{x | f(x) = y\}$.

It is easy to see that each element $x \in X_y$ is derived from y by *flipping* an *even* number of its 1-s, 2-s, and so on, where flipping means changing the number $j \in A$ to $j + n \in B$. Since each such set of flippings results in a unique x, all we want to count is the number of flippings. We can flip exactly $0, 2, 4, \ldots$ of the 1-s, so that results in $\binom{l_1}{0} + \binom{l_1}{2} + \binom{l_1}{4} + \cdots = 2^{l_1 - 1}$ flippings. Combine each of them with the $2^{l_2 - 1}$, $2^{l_3 - 1}$, etc. ways of flipping the 2-s, 3-s etc. respectively to get the total number of flippings: $2^{l_1 - 1} 2^{l_2 - 1} \cdots 2^{l_n - 1} = 2^{l_1 + l_2 + \cdots + l_n - n} = 2^{k - n}$. This shows that $|X_y| = 2^{k - n}$ and the proof is complete. --Vbarzov 03:04, 5 September 2008 (UTC)