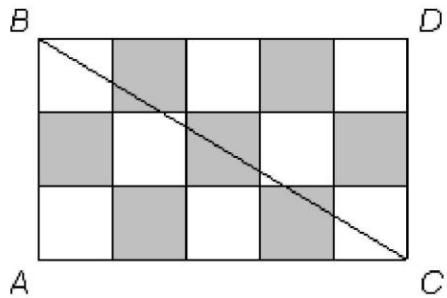
IMO 1997 Solutions

[Note: From the Post-IMO97 Web Page in Argentina]

Problem 1

(a) Let ABC be a right-angled triangle whose vertices have integer coordinates and whose legs lie along edges of the squares with $<A=90^{o}$, $AB=m$ and $AC=n$. Let us consider the mxn rectangle $ABCD$ as shown in the figure.



For an arbitrary polygon P let us denote by $S_1(P)$ the total area of the black part of P and by $S_2(P)$ the total area of its white part.

When m and n are of the same parity the colouring of the rectangle $ABCD$ is centrally symmetric about the midpoint of the hypotenuse BC. Hence $S_1(ABC) = S_1(BCD)$ and $S_2(ABC) = S_1(BCD)$. Therefore

$$f(m, n) = | S_1(ABC) - S_2(ABC) | = 1/2 \, | S_1(ABCD) - S_2(ABCD) |.$$

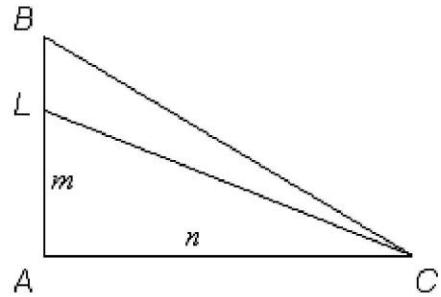
Hence $f(m, n) = 0$ for m, n both even and $f(m, n) = 1/2$ for m, n both odd.

(b) If m, n are both even or both odd the results follows from (a). Suppose now that m is odd and n is even. Consider a point L on AB such that $AL = m - 1$ as shown below.

Since $m - 1$ is even we have $f(m - 1, n) = 0$, i.e., $S_1(ALC) = S_2(ALC)$. Therefore

$$f(m, n) = | S_1(ABC) - S_2(ABC) | = | S_1(LBC) - S_2(LBC) |$$

$$\leq \text{Area } LBC = n/2 \leq 1/2 \max \{m, n\}$$



(c) Let us compute $f(2k + 1,2k)$. As in (b) we will consider a point L on AB such that $AL = 2k$. As $f(2k, 2k) = 0$ and $S_1(LBC) = S_2(ALC)$ we have

$$f(2k + 1,2k) = | S_1(LBC) - S_2(LBC) |.$$

The area of the triangle LBC is k. Suppose without loss of generality that the diagonal LC is all black (see figure below). Then the white part of LBC consists of several triangles BLN_{2k}, $M_{2k-1}L_{2k-1}N_{2k-1}$, ... , $M_1L_1N_1$, each of them being similar to BAC. Their total area is

$$S_2(LBC) = \frac{1}{2}\frac{2k}{2k+1}\left(\left(\frac{2k}{2k}\right)^2 + \left(\frac{2k-1}{2k}\right)^2 + \ldots + \left(\frac{1}{2k}\right)^2\right)$$

$$= \frac{1}{4k(2k+1)}\left(1^2 + 2^2 + \ldots + (2k)^2\right) = \frac{4k+1}{12}.$$

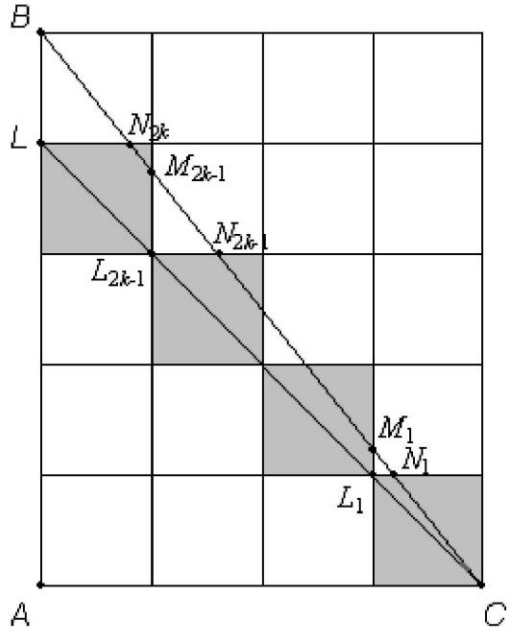
Therefore

$$S_1(LBC) = k - \frac{1}{12}(4k+1) = \frac{1}{12}(8k-1)$$

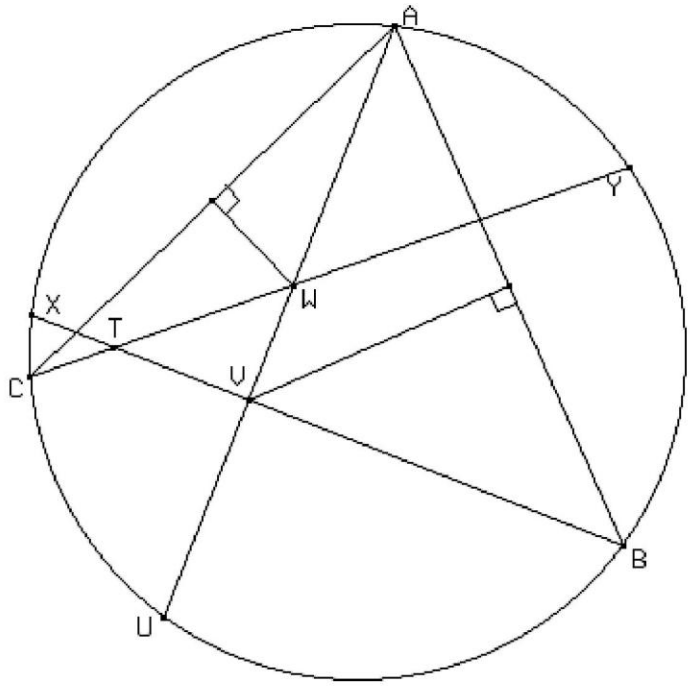
and thus

$$f(2k+1, 2k) = \frac{2k-1}{6}$$

This function takes arbitrarily large values.



Problem 2



Let us denote the diameters which are perpendicular bisectors of AB and AC by b and c respectively. Let us fix an orientation for arcs. Then the notation $\hat{}PQ$ defines a unique arc on the circle.

Let X, Y be the second intersections of the circle with the lines BV and CW respectively. Then the chord CY is the mirror image of the chord AU in c.

Thus $\hat{}YC = \hat{}AU$. Similarly $\hat{}XB = \hat{}AU$. Then $\hat{}XB = \hat{}YC$ and hence the line segments BX and YC are symmetric with respect to the diameter d which passes through the midpoint of BY, with X

being the mirror image of C and Y being the mirror image of B.

As the point T lies on the line d, the perpendicular bisector of both BY and CX, we have $TB = TY$ and $TC = TX$.

Then $AU = BX = BT + TX = BT + CT$.

Problem 3

For any permutation $\pi = (y_1, y_2, \ldots, y_n)$ of $(x_1, x_2, \ldots, x_n)$ denote by $S(\pi)$ the value of the sum $y_1 + 2y_2 + \ldots + ny_n$. Let $r = (n + 1)/2$. It has to be shown that $|S(\pi)| \leq r$ for some permutation π.

Let π_0 be the identity permutation $\pi_0 = (x_1, \ldots, x_n)$ and let $\tilde{\pi}$ be the reverse permutation $\tilde{\pi} = (x_n, \ldots, x_1)$. If $|S(\pi_0)| \leq r$ or $|S(\tilde{\pi})| \leq r$, the claim is true. Assume for the sequel that $|S(\pi_0)| > r$ and $|S(\tilde{\pi})| > r$. Note that

$$S(\pi_0) + S(\tilde{\pi}) = (x_1 + 2x_2 + \ldots + nx_n) + (x_n + 2x_{n-1} + \ldots + nx_1) = (n + 1)(x_1 + \ldots + x_n),$$

and hence $|S(\pi_0) + S(\tilde{\pi})| = n + 1 = 2r$. Since each one of the numbers $S(\pi_0)$ and $S(\tilde{\pi})$ exceeds r in absolute value, they must have opposite signs. Accordingly, one of them is grater than r and the other one is smaller than $-r$.

Starting from π_0, we are able to obtain any permutation by successive transpositions of neighbouring elements. In particular, there exists a chain of permutations $\pi_0, \pi_1, \ldots, \pi_m$ such that $\pi_m = \tilde{\pi}$ and, for each $i \in \{0, \ldots, m-1\}$, the permutation π_{i+1} arises from π_i by interchanging two of its neighbouring terms.

This means that if $\pi_i = (y_1, \ldots, y_n)$ and $\pi_{i+1} = (z_1, \ldots, z_n)$, then there is an index $k \in \{1, \ldots, n - 1\}$ such that

$$z_k = y_{k+1}, \quad z_{k+1} = y_k; \qquad z_j = y_j \quad \text{for } j \neq k, k + 1.$$

And since the numbers x_i do not exceed r in absolute value,

$$\begin{aligned} |S(\pi_{i+1}) - S(\pi_i)| &= |kz_k + (k + 1)z_{k+1} - ky_k (k + 1)y_{k+1}| \\ &= |y_k - y_{k+1}| \leq |y_k| + |y_{k+1}| \leq 2r. \end{aligned}$$

This shows that the distance between any two successive numbers in the sequence $S(\pi_0), S(\pi_i), \ldots, S(\pi_m)$ is not greater than $2r$.

Recall that the numbers $S(\pi_0)$ and $S(\pi_m)$, regarded as points of the real line, lie putside the interval $[-r, r]$, on distinct sides of it. It follows that at least one of the numbers $S(\pi_i)$ has to hit an interval. Thus we have $S(\pi_i) \leq r$ for a certain π_i, and the assertion is proved.

Problem 4

(a) Let $n > 1$ be an integer. Suppose that a silver n x n matrix A exists. Let x be a fixed element of $\{1, 2, \ldots, 2n-1\}$ which does not appear on the diagonal of A. (Such an element exists, since there are only n elements on the diagonal but $2n - 1$ elements in total.) The union of the ith row and the ith column we shall call it ith cross. The element x appears in each cross exactly once. If x is the (i, j)th entry of A, then it belongs to the ith cross and also to the jth cross. In this case we say that these crosses are x-linked. (For example, in the matrix A below the 1st and the 4th crosses are 6-linked.) This implies that all n crosses are partitioned into pairs of x-linked ones, and therefore n is even. But 1997 is an odd number.

(b) For $n = 2$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

is a silver matrix. For $n = 4$ we have already several of them, for example

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 1 & 7 & 5 \\ 4 & 6 & 1 & 2 \\ 7 & 4 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 6 & 7 \\ 7 & 5 & 1 & 3 \\ 6 & 4 & 2 & 1 \end{bmatrix}$$

The construction of A can be generalized. Suppose that an n x n silver matrix exists. Then $2n$ x $2n$ silver matrix D can be constructed as follows:

$$D = \begin{bmatrix} A & B \\ C & A \end{bmatrix}$$

where B is the n x n matrix obtained from A by adding $2n$ to each entry, and C is the matrix obtained from A by adding $2n$ to each entry, and C is the matrix obtained from B by replacing all its diagonal entries by $2n$. This matrix will be again a silver matrix. To prove this let us consider the ith cross of D. Suppose that $i \leq n$; the other case is similar. This cross is composed of the ith cross of A, the ith row of B and the ith column of C. The ith cross of A contains numbers $\{1, 2, \ldots, 2n-1\}$. The ith row of B and the ith column of C together contain the numbers $\{2n, 2n+1, \ldots, 4n-1\}$.

Problem 5

Let a, b be a solution, let d be the greatest common divisor of a and b and let $a = du$, $b = dv$, with integers u, v being coprime, The given equation is equivalent to

$$(du)^{dv^2} = (dv)^u. \tag{1}$$

Comparing the exponents dv^2 and u in (1), we distinguish three cases.

Case 1. $dv^2 = u$. Then (1) implies $u = v$. Since u, v are coprime, we get $u = v = 1$; and $dv^2 = u$ yields $d = 1$. Hence $a = b = 1$, which is a solution.

Case 2. $dv^2 > u$. Rewriting (1) as

$$d^{dv^2-u} \,.\, u^{dv^2} = v^u, \tag{2}$$

we see that u^{dv^2} divides v^u. Since u, v are coprime, $u = 1$ and (2) becomes

$$d^{dv^2-1} = v. \tag{3}$$

Now, if $d = 1$ then by (3) $v = 1$ and the inequality $dv^2 > u$ fails to hold. And for $d \geq 2$ we have

$$d^{dv^2-1} \geq 2^{2v^2-1} \geq 2^{2v-1} > v \quad \text{for } v = 1, 2, 3, \ldots ;$$

The last inequality is easily verified by induction on v. This contradicts (3) and shows that there are no solutions in this case.

Case 3. $dv^2 < u$. (Hence $u > d$.) Rewriting (1) as

$$u^{dv^2} = d^{u-dv^2} \,.\, v^u, \tag{4}$$

We see that v^u divides u^{dv^2}. Since u, v are coprime, $v = 1$ and (4) becomes

$$u^d = d^{u-d}. \tag{5}$$

The exponentiation bases in (5) satisfy the inequality $u > d$; hence the exponents satisfy the opposite inequality $d < u - d$.

By (5), any prime divisor p is of d is also a divisor of u. Taking y, z to be the greatest exponents with $p^y \mid u$, $p^z \mid d$, we obtain from (5) $yd = z(u - d)$, which shows that $y > z$. Thus d divides u, so $u = kd$ or a certain integer k. Note that $k \geq 3$ because $u>2d$. Substituting $u = kd$ into (5), we get

$$k = d^{k-2}. \tag{6}$$

It follows that d cannot be 1. Therefore $d \geq 2$.

If $k = 3$, then by (6) $d = 3$, whence $u = 9$, $a = 27$, $b = 3$.
If $k = 4$, then by (6) $d^2 = 4$, whence $d = 2$, $u = 8$, $a = 16$, $b = 2$.
If $k \geq 5$, then $d^{k-2} \geq 2^{k-2} > k$ (by easy induction) and (6) fails.

Thus, the equation has three solutions: $(a, b) = (1, 1)$, $(27, 3)$ and $(16, 2)$.

Problem 6

If $n = 2k + 1$ is any odd integer greater than 1, then every representation of n in the given form has a "1" as one of its summands. Deleting we obtain a representation of the number $2k$. And conversely, by attaching a "1" to a representation of $2k$ we obtain a representation of $2k + 1$. The correspondence that results is obviously bijective. The recursion formula follows:

$$f(2k + 1) = f(2k). \tag{1}$$

Further, if $n = 2k$ is any positive even integer, then every representation of n in the given form is one of the following two types: either it contains some terms equal to 1 or it does not contain such terms. In the first case we can delete one "1" to obtain a representation of $2k - 1$: as before, we see that there is a bijection between the "first-type" representations of $2k$ and all representations of $2k - 1$. In the case of a "second-type" representation (with no terms equal to 1), we can divide all its terms by 2 and obtain a representation of k; and also this correpondence is bijective. The second recursion formula follows:

$$f(2k) = f(2k - 1) + f(k). \tag{2}$$

Both of these formulae hold for any integer $k \geq 1$. Obviously, $f(1) = 1$. Let us additionally define $f(0) = 1$; then the formula (1) holds for $k = 0$ as well. It follows from (1) and (2) that the function f is non-decreasing.

According to (1), the number $f(2k - 1)$ in the formula (2) can be replaced by $f(2k - 2)$, yielding the equation

$$f(2k) - f(2k - 2) = f(k) \qquad \text{for } k = 1, 2, 3, \ldots .$$

Taking any integer $n \geq 1$ and summing these equations up over $k = 1, \ldots, n$, we obtain the following formula:

$$f(2n) = f(0) + f(1) + \ldots + f(n) \qquad \text{for } n = 1, 2, 3, \ldots . \tag{3}$$

We pass the two-sided estimation of the sequence $(f(2^n): n = 1, 2, 3, \ldots)$. The upper estimate is easy: in the formula (3) all the summands are not greater than the last one. And since $2 = f(2) \leq f(n)$ for $n \geq 2$, we get

$$f(2n) = 2 + (f(2) + \ldots + f(n)) \leq 2 + (n - 1) f(n)$$

$$\leq f(n) + (n - 1) f(n) = nf(n) \qquad \text{for } n = 2, 3, 4, \ldots .$$

Consequently,

$$f(2^n) \leq 2^{n-1} . f(2^{n-1}) \leq 2^{n-1} . 2^{n-2} . f(2^{n-2}) \leq 2^{n-1} . 2^{n-2} . 2^{n-3} . f(2^{n-3})$$

$$\leq \ldots \leq 2^{(n-1) + (n-2) + \ldots + 1} . f(2) = 2^{n(n-1)/2} . 2.$$

And since $2^{n(n-1)/2} . 2 < 2^{n^2/2}$ for $n \geq 3$, the upper estimate results.

To work out the lower estimate, we first show that

$$f(b + 1) - f(b) \geq f(a + 1) - f(a) \tag{4}$$

whenever $b \geq a \geq 0$ are integers of the same party. Indeed: if a and b are both even, then in view of (1) we have zeros on both sides of (4); and if a and b are odd, then by (2) we get $f(b + 1) - f(b) = f((b + 1)/2)$, $f(a + 1) - f(a) = f((a + 1)/2)$ and the inequality in (4) holds because the function f is non-decreasing.

Let us take any integers $r \geq k \geq 1$, r even, and substitute subsequently in (4) the numbers $a = r - j$, $b = r + j$ for $j = 0, \ldots, k - 1$. Then add the inequalities that result, to obtain

$$f(r + k) - f(r) \geq f(r + 1) - f(r - k + 1).$$

Since r is even, we have $f(r + 1) = f(r)$, and hence

$$f(r + k) + f(r - k + 1) \geq 2 f(r) \qquad \text{for } k = 1, \ldots, r..$$

Summing these inequalities up over $k = 1, \ldots, r$ we get

$$f(1) + f(2) + \ldots + f(2r) \geq 2r f(r).$$

In view of (3) the sum on the left-hand side is equal to $f(4r) - 1$. Thus

$$f(4r) \geq 2r f(r) + 1 > 2r f(r) \qquad \text{for every integer } r \geq 2.$$

Taking $r = 2^{m-2}$ we hence obtain

$$f(2^m) > 2^{m-1} f(2^{m-2}). \tag{5}$$

To ensure that $r = 2^{m-2}$ is even, m ought to be an integer greater than 2; note, however, that (5) is also true for $m = 2$.

Finally, let n be any integer greater than 1. If l is a positive integer such that $2l \leq n$, then applying the inequality (5) for $m = n, n-1, \ldots, n - 2l + 2$ we get

$$f(2^n) > 2^{n-1} . f(2^{n-2}) > 2^{n-1} . 2^{n-3} . f(2^{n-4}) > 2^{n-1} . 2^{n-3} . 2^{n-5} . f(2^{n-6})$$

$$> \ldots > 2^{(n-1) + (n-3) + \ldots + (n - 2l + 1)} \,.\, f(2^{n-2l}) = 2^{l\,(n - l)} \,.\, f(2^{n - 2l}).$$

Now, if n is even, take $l = n/2$; and if n is odd, take $l = (n - 1)/2$. The resulting inequalities are:

$$f(2^n) > 2^{n^2/4} \,.\, f(2^0) = 2^{n^2/4} \qquad \text{for } n \text{ even,}$$

$$f(2^n) > 2^{(n^2 - 1)/4} \,.\, f(2^1) = 2^{(n^2 - 1)/4} \,.\, 2 > 2^{n^2/4} \qquad \text{for } n \text{ odd.}$$

We have obtained the required lower estimate for every integer $n \geq 2$. (It is also valid for $n = 1$, as can be verified directly.)